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INFINITE TYPE POWER SERIES SUBSPACES OF INFINITE TYPE POWER SERIES SPACES†

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ABSTRACT

We consider the problem of finding all power series subspaces of a given infinite type power series space. A necessary condition is obtained which is similar to a property of complemented subspaces of nuclear Fréchet spaces. In some cases, a complete solution is obtained and this leads to new information on a conjecture of C. Bessaga.

This paper is a continuation of the investigations begun in [3]. The main problem which we consider here is to find, for a given nuclear exponent sequence α of infinite type, necessary and sufficient conditions on β so that $\Lambda_\infty(\beta)$ is isomorphic to a subspace of $\Lambda_\infty(\alpha)$. In view of a well-known result of V. P. Zaharjuta [6], this is the same as finding all power series subspaces of $\Lambda_\infty(\alpha)$. Theorem 1 is a technical characterization of such β which is useful but is not completely satisfactory. Theorem 2 gives a necessary condition on β to the effect that, up to isomorphism, β must be obtainable from the sequence α by deleting some terms and repeating others finitely many times. This is similar to a property of complemented subspaces of nuclear Fréchet spaces (see [1], Theorem 2.2). It is shown that the condition of Theorem 2, even when strengthened by the additional requirement (easily established as necessary from considerations of diametric dimension) that $\sup_n(\alpha_n/\beta_n)<\infty$, is not sufficient. On the other hand, it is shown in Theorem 3 that if $\lim_n(\alpha_{n+1}/\alpha_n)=\infty$ or $\sup_n(\alpha_{2n}/\alpha_n)<\infty$ then the necessary condition of Theorem 2 can be strengthened so as to exclude repetitions and it is then sufficient. This permits us to establish the following conjecture of C. Bessaga [1] for the case $X=\Lambda_\infty(\alpha)$ and α satisfies either of the above two conditions: *if X is a nuclear Fréchet space with a basis (x_n) then Y is isomorphic to a complemented subspace of X and has a basis if and only if Y is isomorphic to the subspace generated by a subsequence of (x_n)*. Up until now this conjecture was known to

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hold for all power series spaces of finite type [4] and those power series spaces of infinite type which are isomorphic to their square (that is, $\sup_n(\alpha_{2n}/\alpha_n) < \infty$).

We do not know if Theorem 3 holds for all α. All of our considerations involve infinite type power series subspaces of an infinite type power series space $\Lambda_\infty(\alpha)$. In view of Zaharjuta's result, this includes all power series subspaces of $\Lambda_\infty(\alpha)$. It can be shown that $\Lambda_\infty(\alpha)$ contains subspaces which are not power series spaces, and in fact, contain no power series subspaces, but we do not know of a simple example of a subspace of $\Lambda_\infty(\alpha)$ which is not a power series space.

Some of our initial results are analogous in statement and proof to those obtained in [3], and one result (Theorem 2) does not appear in [3], but its analogue is true.

Definitions and terminology not explicitly explained here can be found in [1] or [3]. By *subspace* we shall mean a closed, infinite dimensional linear subspace. The symbol $\mathbf{N}$ will denote the sequence of positive integers, sometimes called the *indices*. We shall say that two positive sequences (α_n) and (β_n) are *asymptotic* and write $(\alpha_n) \sim (\beta_n)$ provided that the sequences (α_n/β_n) and (β_n/α_n) are bounded above. In the nuclear power series space $\Lambda_\infty(\alpha)$, we shall indicate by $(\|\cdot\|_k)$ the fundamental sequence of seminorms given by

$$\|\xi\|_k = \sup_i |\xi_i| k^{\alpha_i}, \qquad k = 1, 2 \cdots .$$

Analogously to [3], we introduce some auxiliary quantities based on the following parameters:

α, β —nuclear exponent sequences of infinite type;

(y_n)—a basic sequence in $\Lambda_\infty(\alpha)$ of the form:

$$y_n = \sum_{i=1}^{\infty} t_i^n e_i, \qquad n = 1, 2, \cdots ;$$

(d_n)—a sequence of non-zero scalars;

π —a permutation of $\mathbf{N}$.

Then we define, for n, $k = 1, 2, \cdots$

$$q_n^k = \max\{q : k^{\alpha_q} |t_q^n| = \max_i k^{\alpha_i} |t_i^n|;$$

$$\gamma_n^k = \frac{\alpha_{q_n^k}}{\beta_{\pi(n)}};$$

$$\mu_n^k = d_n^{1/\beta_{\pi(n)}} |t_{q_n^k}^n|^{1/\beta_{\pi(n)}} k^{\gamma_n^k};$$

$Y =$ subspace of $\Lambda_\infty(\alpha)$ generated by (y_n).

Using the same argument as in Lemma 3 of [3] we obtain the inequality,

$$\left(\frac{k+1}{k}\right)^{\gamma_n^k} \leqq \frac{\mu_n^{k+1}}{\mu_n^k} \leqq \left(\frac{k+1}{k}\right)^{\gamma_n^{k+1}} \qquad \text{for all } n,\ k. \tag{1}$$

We are then able to prove

THEOREM 1. *In the context of the preceding notation, the following are equivalent.*

(i) *Y is isomorphic to $\Lambda_\infty(\beta)$;*

(ii) *There exist (d_n) and π such that for each k, $\overline{\lim}_n \mu_n^k < \infty$ and $\sup_k \underline{\lim}_n \mu_n^k = \infty$;*

(iii) *There exists π such that for each k, $\sup_n \gamma_n^k < \infty$ and $\sup_k \underline{\lim}_n \sum_{j=1}^k (\gamma_n^j / j) = \infty$.*

The proof of this theorem is based on inequality (1) and is very similar to the proof of Theorem 1 of [3]. The only significant change is that because of the different form of (1), we must here replace the use of the quantity $\log((j^2+2j+1)/(j^2+2j))$ by the use of the quantity $\log((j+1)/j)$. This leads to the appearance of j in the denominator in condition (iii) above as opposed to the j^2 which appears in the corresponding position in [3]. We may note here that the referee has pointed out that in [3], page 265, line 13, d_n should be replaced by d_n^{-1}.

We also obtain the following result using an argument identical to the proof of Proposition 2 of [3].

COROLLARY 1. *If Y is isomorphic to $\Lambda_\infty(\beta)$ and π is chosen so that condition (iii) of Theorem 1 holds, then there exists k_0 such that for $k \geqq k_0$ we have*

$$\underline{\lim}_n \gamma_n^k > 0 \quad \text{and} \quad \lim_n q_n^k = \infty.$$

The next result does not appear in [3], but the same argument leads to an analogous conclusion for infinite type power series subspaces of finite type power series spaces.

THEOREM 2. *If $\Lambda_\infty(\beta)$ is isomorphic to a subspace of $\Lambda_\infty(\alpha)$ then there exists a non-decreasing unbounded sequence of indices (i_n) such that $\beta \sim (\alpha_{i_n})_n$.*

PROOF. The hypothesis means that there is a basic sequence (y_n) in $\Lambda_\infty(\alpha)$ such that Y is isomorphic to $\Lambda_\infty(\beta)$. We choose π according to (iii) of Theorem

1 and apply Corollary 1 to conclude that $(\alpha_{q_n^{k_0}})_n \sim (\beta_{\pi(n)})$. Also by Corollary 1 we can find a permutation σ of N such that if $i_n = q_{\sigma(n)}^{k_0}$, then (i_n) is non-decreasing and unbounded. Thus it follows that the map $(\xi_n) \rightsquigarrow (\xi_{\pi\sigma(n)})$ defines an isomorphism of $\Lambda_\infty(\beta)$ onto $\Lambda_\infty((\alpha_{i_n}))$. The conclusion follows from elementary properties of power series spaces.

REMARK. If we assume that $\Lambda_\infty(\beta)$ is complemented in $\Lambda_\infty(\alpha)$, then Theorem 2 is a special case of Theorem 2.2 of [1]. Thus we obtain the same conclusion without assuming that our subspace is complemented, but requiring instead that it is a power series space.

Theorem 2 gives a necessary condition on β for $\Lambda_\infty(\alpha)$ to have a subspace isomorphic to $\Lambda_\infty(\beta)$. It is interesting to compare it with the condition obtained from considerations of diametric dimension. Applying the results of [1], 1.10 we conclude easily that if $\Lambda_\infty(\beta)$ is isomorphic to a subspace of $\Lambda_\infty(\alpha)$, then $\sup_n (\alpha_n/\beta_n) < \infty$. Again this condition is not comparable with ours. For example, if $\alpha_n = 2^{2^n}$ and we set $\beta_n = \alpha_{[n/2]+1}$, then our condition holds but the diametric dimension condition does not. The reverse is true if we set $\beta_n = (1/2)(\alpha_n + \alpha_{n+1})$. Both conditions hold if we set $\beta_n = \alpha_{[n/2]+2}$. The remainder of this paper is devoted to a closer analysis of special cases leading to the conclusion that if $\beta_n = \alpha_{[n/2]+2}$, then $\Lambda_\infty(\beta)$ is *not* isomorphic to a subspace of $\Lambda_\infty(\alpha)$. In particular, it will follow that *neither of the two necessary conditions (nor their union) is sufficient.*

PROPOSITION 1. *Let F be a subspace of $\Lambda_\infty(\alpha)$ such that F is isomorphic to a power series space of infinite type and suppose that $\lim(\alpha_{n+1}/\alpha_n) = \infty$. Then any basis for F has a permutation (y_n) for which there exists a non-decreasing unbounded sequence of indices (i_n) and k_0 such that*

$$\forall k \geqq k_0 \, \exists n_k \ni q_n^k = i_n \quad \text{for} \quad n \geqq n_k.$$

PROOF. We begin with a basic sequence in $\Lambda_\infty(\alpha)$ and since we are permitted one permutation we choose (y_n) so that Y is isomorphic to some $\Lambda_\infty(\beta)$ for some β and π of Theorem 1 is the identity.

Arguing exactly as in the proof of Theorem 2 we obtain (i_n) and k_0 so that $(\alpha_{q_n^k})_n \sim (\beta_{\pi(n)})_n = \beta$ for each $k \geqq k_0$. By the conclusion of Theorem 2 it follows that $(\alpha_{q_n^k})_n \sim (\alpha_{i_n})$ for each $k \geqq k_0$ and the hypothesis that $\lim(\alpha_{n+1}/\alpha_n) = \infty$ implies that $q_n^k = i_n$ for n sufficiently large and this is the desired conclusion.

It will be convenient to introduce some notation in the context of Proposition 1. We take (ν_n) to be the strictly increasing sequence whose range is identical to that of (i_n). Then we can take $0 = p_0 < p_1 < p_2 < \cdots$ and write

$$i_j = \nu_n \quad \text{for} \quad p_{n-1} < j \leqq p_n, \qquad n = 1, 2, \cdots$$

and

$$q_j^k = i_j = \nu_n \quad \text{for} \quad p_{n-1} < j \leqq p_n, \quad n \geqq n_k.$$

We will take $\beta_n = \alpha_{i_n}$ and it follows that F is isomorphic to $\Lambda_\infty(\beta)$.
In the sequel, this notation will be referred to as the "context of Proposition 1 ".

PROPOSITION 2. *In the context of Proposition* 1, *there is an* m_0 *such that* $i_n \geqq n$ *and* $p_n \leqq \nu_n$ *for* $n > m_0$.

PROOF. As we have seen, it follows from considerations of diametric dimension that since $\Lambda_\infty(\beta)$ is isomorphic to a subspace of $\Lambda_\infty(\alpha)$, then $\sup_n (\alpha_n / \beta_n) < \infty$. Hence $\sup_n (\alpha_n / \alpha_{i_n}) < \infty$ and since $\lim (\alpha_{n+1}/\alpha_n) = \infty$ it follows that $n \leqq i_n$ for n sufficiently large. In particular, for n sufficiently large, $p_n \leqq i_{p_n} = \nu_n$.

The following lemma is a variation of a result of Bourbaki, [2] II §2, Lemme 1. The proof is straightforward, one simply makes, for each d, the same construction as given by Bourbaki.

LEMMA. *Let F be a subspace of a locally convex space E and let* $(V_d)_{d \in D}$ *be a fundamental system of absolutely convex open neighborhoods of* 0 *for F such that E has a fundamental system of neighborhoods of* 0 *indexed by D. Then there is a fundamental system of absolutely convex open neighborhoods of* 0, $(U_d)_{d \in D}$ *for E such that*

$$V_d = U_d \cap F, \; d \in D.$$

In the next result, we use a standard theorem on stability to replace our basic sequence (y_n) by an equivalent sequence having a special form.

PROPOSITION 3. *In the context of Proposition* 1, *we set*

$$\bar{y}_j = \sum_{i=1}^{\nu_n} t_i^j e_i, \; p_{n-1} < j \leqq p_n \quad \textit{where} \quad y_j = \sum_{i=1}^{\infty} t_i^j e_i, \quad j = 1, 2, \cdots.$$

Then there exists j_1 *such that* $(\bar{y}_j)_{j=j_1}^{\infty}$ *is a basic sequence in* $\Lambda_\infty(\alpha)$ *equivalent to* $(y_j)_{j=j_1}^{\infty}$.

PROOF. Since α is a nuclear exponent sequence of infinite type, we have $C = \sup (\log n / \alpha_n) < \infty$. Let $C_1 = e^C$. Also we have $\bar{k}$ such that $\Sigma \bar{k}^{-\alpha_n} < \infty$.

Since (y_n) is a basic sequence in $\Lambda_\infty(\alpha)$ which generates a subspace Y isomorphic to $\Lambda_\infty(\beta)$, it follows that there is a fundamental system of norms for Y such that (y_n) is a basis in the completion of Y with respect to each of these norms. By the lemma, these norms can be extended to a fundamental system of norms $(\|\cdot\|_k')$ for $\Lambda_\infty(\alpha)$. Thus (y_n) is a basic sequence in each completion of $\Lambda_\infty(\alpha)$ with respect to $\|\cdot\|_k'$. Also, we have subsequences of indices (k_μ), (l_μ) such that $k_1 \geqq k_0$ and

$$\|x\|_{k_\mu} \leqq \|x\|'_{l_\mu} \leqq \|x\|_{k_{\mu+1}}, \; x \in \Lambda_\infty(\alpha), \quad \mu = 1, 2, \cdots .$$

Fix $\mu = 1, 2, \cdots$ and let $\bar{k} \geqq (c_1 \bar{k} k_{\mu+1}/k_\mu)$. Choose $n_0 \geqq n_{k_\mu}, n_{\bar{k}k_{\mu+1}}$(where these quantities are obtained from Proposition 1) such that

$$\alpha_{\nu_n+1} - \alpha_{\nu_n} \geqq \alpha_{\nu_n} \quad \text{for} \quad n \geqq n_0.$$

The last inequality is possible because $\lim(\alpha_{n+1}/\alpha_n) = \infty$ and (ν_n) is strictly increasing. Thus we have $q_j^k = \nu_n$ for $p_{n-1} < j \leqq p_n$ and $n \geqq n_k$ so we can apply this for $k = k_\mu$ and $k = \bar{k} k_{\mu+1}$ to obtain,

$$\|y_j\|_{k_\mu} = k_\mu^{\alpha_{\nu_n}} |t_{\nu_n}^j| \quad \text{for} \quad p_{n-1} < j \leqq p_n, \qquad n \geqq n_0,$$
$$\|y_j\|_{\bar{k}k_{\mu+1}} = (\bar{k} k_{\mu+1})^{\alpha_{\nu_n}} |t_{\nu_n}^j| \quad \text{for} \quad p_{n-1} < j \leqq p_n, \quad n \geqq n_0.$$

The second statement along with the definition of the norm implies that

$$(\bar{k}k_{\mu+1})^{\alpha_i} |t_i^j| < (\bar{k}k_{\mu+1})^{\alpha_{\nu_n}} |t_{\nu_n}^j| \quad \text{for} \quad i > \nu_n, \; p_{n-1} < j \leqq p_n, \qquad n \geqq n_0.$$

Hence

$$\begin{aligned}
\|y_j - \tilde{y}_j\|_{k_{\mu+1}} &= \sup_{i > \nu_n} (k_{\mu+1})^{\alpha_i} |t_i^j| \quad \text{for} \quad p_{n-1} < j \leqq p_n \\
&\leqq \frac{(\bar{k}k_{\mu+1})^{\alpha_{\nu_n}}}{\bar{k}^{\alpha_{\nu_n+1}}} |t_{\nu_n}^j| \quad \text{for} \quad p_{n-1} < j \leqq p_n, \quad n \geqq n_0 \\
&= \frac{1}{\bar{k}^{\alpha_{\nu_n+1} - \alpha_{\nu_n}}} \left(\frac{k_{\mu+1}}{k_\mu}\right)^{\alpha_{\nu_n}} \|y_j\|_{k_\mu} \quad \text{for} \quad p_{n-1} < j \leqq p_n, \quad n \geqq n_0 \\
&\leqq \left(\frac{1}{C_1 \bar{k}}\right)^{\alpha_{\nu_n}} \|y_j\|_{k_\mu} \quad \text{for} \quad p_{n-1} < j \leqq p_n, \quad n \geqq n_0.
\end{aligned}$$

Therefore we may compute

$$\sum_{j=1}^{\infty}\frac{\|y_j-\tilde{y}_j\|'_{l_\mu}}{\|y_j\|'_{l_\mu}}\leqq\sum_{j=1}^{\infty}\frac{\|y_j-\tilde{y}_j\|_{k_{\mu+1}}}{\|y_j\|_{k_\mu}}=\sum_{n=1}^{\infty}\sum_{j=p_{n-1}+1}^{p_n}\frac{\|y_j-\tilde{y}_j\|_{k_{\mu+1}}}{\|y_j\|_{k_\mu}}$$

$$\leqq\bar{C}_\mu\sum_{n=1}^{\infty}\frac{p_n-p_{n-1}}{(C_1k)^{\alpha_{\nu_n}}}\leqq\bar{C}_\mu\sum_{n=1}^{\infty}\frac{\nu_n}{e^{C\alpha_{\nu_n}}}\bar{k}^{-\alpha_{\nu_n}}$$

$$\leqq\bar{C}_\mu\sum_{n=1}^{\infty}\frac{\nu_n}{e^{\log\gamma_n}}\bar{k}^{-\alpha_{\nu_n}}=\bar{C}_\mu\sum_{n=1}^{\infty}\bar{k}^{-\alpha_{\nu_n}}<\infty.$$

Thus we may apply standard stability results ([5] Cor. 10.1 and Th. 3.1) to conclude that for each μ, there exists j_μ such that $(\tilde{y}_j)_{j=j_\mu}^{\infty}$ is a basic sequence in the completion of $\Lambda_\infty(\alpha)$ with respect to $\|\cdot\|'_{l_\mu}$, equivalent to $(y_j)_{j=j_\mu}^{\infty}$. Since each $\|\cdot\|'_{l_\mu}$ is a norm it follows immediately from standard arguments that $(\tilde{y}_j)_{j=j_1}^{\infty}$ is a basic sequence in $\Lambda_\infty(\alpha)$ equivalent to $(y_i)_{j=j_1}^{\infty}$.

Our next step is to derive an inequality involving ν_n and p_n.

PROPOSITION 4. *In the context of Proposition* 1, *it follows that for each* $k\geqq k_0$ *there is a* $\bar{k}$, C_k *and* $\tilde{n}_k$ *such that*

$$k^{\alpha_{\nu_n}}\leqq C_k\bar{k}^{\alpha_{\nu_n}-(p_n-p_{n-1})+1}(p_n-p_{n-1}),\qquad n\geqq\tilde{n}_k.$$

PROOF. Let $(\tilde{y}_j)$ be as in Proposition 3 and let G be the subspace of $\Lambda_\infty(\alpha)$ generated by $(\tilde{y}_j)_{j=j_1}^{\infty}$. For each $j\geqq j_1$, let $A_j\colon G\to G$ be the projection determined by the basis $(\tilde{y}_j)_{j=j_1}^{\infty}$ onto the one dimensional subspace generated by y_j. Then from standard basis theory it follows that for each k there is $\bar{k}$ and $\bar{C}_k$ such that

$$\|A_jy\|_k\leqq\bar{C}_k\|y\|_{\bar{k}},\ y\in G,\quad j\geqq j_1.$$

Choose $n\geqq n_k$, n_{k_0} and such that $p_{n-1}\geqq j_1$ and consider the $\nu_n\times(p_n-p_{n-1})$ matrix (t_i^j) where $1\leqq i\leqq\nu_n$ and $p_{n-1}<j\leqq p_n$. We consider this matrix as defining a map from $K^{p_n-p_{n-1}}$ to K^{ν_n} ($K=$ the field of scalars). Since $(\tilde{y}_j)_{j=j_1}^{\infty}$ is a basic sequence, $(\tilde{y}_j)_{p_{n-1}<j\leqq p_n}$ is linearly independent so this matrix has rank p_n-p_{n-1} and the range of the map is a p_n-p_{n-1} dimensional subspace H_1 of K^{ν_n}. If H_2 is the subspace of K^{ν_n} consisting of all vectors whose last $p_n-p_{n-1}-1$ coordinates vanish then it follows from considerations of dimension that $H_1\cap H_2$ contains a non-zero vector. This means that there are scalars ξ_j, $p_{n-1}<j\leqq p_n$ not all zero such that

$$\sum_{j=p_{n-1}+1}^{p_n} \xi_j t_i^j = 0 \quad \text{for} \quad \nu_n - (p_n - p_{n-1}) + 1 < i \leqq \nu_n.$$

Thus we can write

$$x = \sum_{j=p_{n-1}+1}^{p_n} \xi_j \tilde{y}_j = \sum_{i=1}^{\nu_n-(p_n-p_{n-1})+1} \sum_{j=p_{n-1}+1}^{p_n} \xi_j t_i^j e_i \in G,$$

and so, for an appropriate choice of i_0, j_0 with $1 \leqq i_0 \leqq \nu_n - (p_n - p_{n-1}) + 1$ and $p_{n-1} < j_0 \leqq p_n$ we have

$$\|x\|_{\bar{k}} = \left| \sum_{j=p_{n-1}+1}^{p_n} \xi_j t_{i_0}^j \right| \bar{k}^{\alpha_{i_0}} \leqq |\xi_{j_0} t_{i_0}^{j_0}| (p_n - p_{n-1}) \bar{k}^{\alpha_{\nu_n-(p_n-p_{n-1})+1}}$$

However, by Proposition 1 since $n \geqq n_{k_0}$,

$$|t_{i_0}^{j_0}| \leqq |t_{i_0}^{j_0}| k_0^{\alpha_{i_0}} \leqq |t_{\nu_n}^{j_0}| k_0^{\alpha_{\nu_n}}$$

and so

$$\|x\|_{\bar{k}} \leqq |\xi_{j_0} t_{\nu_n}^{j_0}| (p_n - p_{n-1}) k_0^{\alpha_{\nu_n}} \bar{k}^{\alpha_{\nu_n-(p_n-p_{n-1})+1}}$$

On the other hand, since $n \geqq n_k$ we have from Proposition 1,

$$\|A_{j_0} x\|_k = |\xi_{j_0}| \, \|\tilde{y}_{j_0}\|_k = |\xi_{j_0}| \, |t_{\nu_n}^{j_0}| k^{\alpha_{\nu_n}} .$$

Hence we have,

$$\left(\frac{k}{k_0}\right)^{\alpha_{\nu_n}} \leqq \bar{C}_k (p_n - p_{n-1}) k^{\alpha_{\nu_n-(p_n-p_{n-1})+1}}, \qquad n \geqq \tilde{n}_k$$

where $\tilde{n}_k$ is chosen so that $n \geqq \tilde{n}_k$ implies $n \geqq n_k$, n_{k_0} and $p_{n-1} \geqq j_1$. Finally, the desired conclusion is obtained by applying the entire arguments to $k k_0$.

This completes the preliminary computations and we are ready to obtain our results for the special case, $\lim(\alpha_{n+1}/\alpha_n) = \infty$.

THEOREM 3. *If* $\lim(\alpha_{n+1}/\alpha_n) = \infty$, *then a subspace of* $\Lambda_\infty(\alpha)$ *is isomorphic to a power series space if and only if it is isomorphic to the subspace of* $\Lambda_\infty(\alpha)$ *generated by a subsequence of the basis* (e_i).

PROOF. One way is obvious so we may assume that we have a subspace of $\Lambda_\infty(\alpha)$ which is isomorphic to a power series space. In view of Zaharjuta's result [6] on the impossibility of embedding finite type power series spaces in infinite type power series spaces, we may assume that the subspace is of infinite type.

Thus we return to the context of Proposition 1 and assume that we have an infinite set $\mathbf{N}_0$ of indices such that $p_n - p_{n-1} \geqq 2$ for $n \in \mathbf{N}_0$. Applying Propositions 2 and 4 with $k = k_0$ we have $\tilde{k}$, C and $\tilde{n}$ such that

$$k_0^{\alpha_{\nu_n}} \leqq C_{\nu_n} \tilde{k}^{\alpha_{\nu_n - (p_n - p_{n-1}) + 1}} \leqq C_{\nu_n} \tilde{k}^{\alpha_{\nu_n - 1}} \text{ for } n \geqq \tilde{n}, n \in \mathbf{N}_0.$$

This implies that we have $C_1 > 0$ and an infinite set of indices $\mathbf{N}_1$ such that

$$k_0^{\alpha_n} \leqq C_1 n \tilde{k}^{\alpha_{n-1}}, \qquad n \in \mathbf{N}_1$$

so that, since α is a nuclear exponent sequence of infinite type, we have

$$\sup_{n \in \mathbf{N}_1} \frac{\alpha_n}{\alpha_{n-1}} \leqq \sup_{n \in \mathbf{N}_1} \left(\frac{\log C_1}{\alpha_{n-1} \log k_0} + \frac{\log n}{\alpha_{n-1} \log k_0} + \frac{\log \tilde{k}}{\log k_0} \right) < \infty$$

which is a contradiction. Thus $p_n - p_{n-1} = 1$ for n sufficiently large. This means that (i_n) is strictly increasing for $n \geqq n_0$. Hence we have, for $n \geqq n_0$,

$$\| y_n \|_k = | t^{n_k}_{q_n} | k^{\alpha_{q_n^k}} = | t^n_{i_n} | k^{\alpha_{i_n}} \quad \text{for} \quad n \geqq n_k$$

which implies that the map $y_n \rightsquigarrow t^n_{i_n} e_{i_n}$, $n \geqq n_0$ defines an isomorphism of the space generated by $(y_n)_{n \geqq n_0}$ onto the space generated by the *subsequence* $(e_{i_n})_{n \geqq n_0}$. Finally we observe that by Proposition 2, if we originally chose $n_0 > m_0$ then $i_n \geqq n \geqq n_0$ for $n \geqq n_0$. This implies that the space generated by $(y_n)_{n=1}^\infty$ is isomorphic to the space generated by the subsequence $(e_1, \cdots, e_{n_0 - 1}, e_{i_{n_0}}, e_{i_{n_0 + 1}}, \cdots)$ and we are finished.

REMARK. Theorem 3 proves the statement made after Theorem 2 above because, as pointed out by Bessaga ([1], p. 318) if $\alpha_n = 2^{2^n}$ (which satisfies the hypothesis of Theorem 3) and $\beta_n = \alpha_{[n/2]+2}$, then $\Lambda_\infty(\beta)$ is not isomorphic to the subspace of $\Lambda_\infty(\alpha)$ generated by any subsequence of (e_n) so by Theorem 3 it is not isomorphic to any subspace of $\Lambda_\infty(\alpha)$.

Theorem 3 also holds if the hypothesis, $\lim_n (\alpha_{n+1}/\alpha_n) = \infty$, is replaced by the very different condition, $\sup_n (\alpha_{2n}/\alpha_n) < \infty$. To see this, we first apply Theorem 2 and then the same argument used by Bessaga [1] in proving Corollary 3.3.

We conclude by relating some of our results to Bessaga's conjecture.

COROLLARY 2. *Bessaga's conjecture holds for* $\Lambda_\infty(\alpha)$ if $\lim_n (\alpha_{n+1}/\alpha_n) = \infty$.

PROOF. If F is a complemented subspace of $\Lambda_\infty(\alpha)$ and has a basis, then we can apply Theorem 2.2 of [1] to conclude that F is isomorphic to a power series space. The conclusion then follows from Theorem 3.

Finally we point out that it is obvious Theorem 3 and the fact that in a nuclear space any subspace generated by a subsequence of a basis is complemented, that *if* $\lim_n (\alpha_{n+1}/\alpha_n) = \infty$ *or* $\sup_n (\alpha_{2n}/\alpha_n) < \infty$ *then a subspace of* $\Lambda_\infty(\alpha)$ *is isomorphic to a power series space if and only if it is isomorphic to a complemented subspace.* It might be interesting to investigate how general this phenomenon is.

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